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Results on Fixed Points for Multivalued Mappings on an Orbitally Complete Metric Space

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ABSTRACT: In this paper, we prove fixed point theorem for multivalued mappings on an orbitally complete metric space which include the results of Achari[1] and Jain and Bohre[5].

Keywords: Multivalued map and orbitally complete metric space.

INTRODUCTION: Kakutani[6] initiated the study of fixed point problems of multivalued functions in 1941 in finite dimensional spaces. It was extended to infinite dimensional Banach spaces by Bohnenblust and Karlin[2] in 1950.

Nadler[9] introduced the notion multivalued contraction mappings in metric spaces. Singh and Dubey[15] extended the result of Kannan to multivalued mappings which was unified by Reich[13]. All these results were generalized by Iseki[4]. Popa[10] obtained a common fixed point theorem for a sequence of multifunction's on a complete metric space which includes the results of Rus [14], Ray[11 & 12] and Wong[17].

Kaneko[7] extended the concepts of weak commutativity and compatibility see Kaneko et al.[8] for single-valued mappings to the setting of single-valued and multi-valued mappings respectively.

Preliminiries: Let (X,d) be a metric space and B(X) be the set of all bounded subset of X.

For any $x \in X$, $A, B \in B(X)$, we write

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

The function δ satisfies

(i) $\delta(A, B) = \delta(B, A) \geq 0$, $\delta(A, B) = 0 \Rightarrow A = B = \{a\}$,

(ii) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for $A, B, C \in B(X)$.

If $A = \{a\}$,
we write $\delta(A, B) = \delta(a, B)$ and furthermore,
if $B = \{b\}$,
we write $\delta(A, B) = \delta(a, b) = d(a, b)$.

Definition 1: A sequence $\{A_n\}$ of sets in B(X) is said to converge to the subset A of X if the following conditions are satisfied:

(i) For each a in A, there is a sequence $\{a_n\}$ such that $a_n \in A_n$ for all n and $a_n \to a$

(ii) For every $\varepsilon > 0$, there is an integer N such that $A_n \subset A_\varepsilon$ for all $n \geq N$, where A_ε is the union of all open spheres with centers in A and radius ε.

The set A is then said to be the limit of the sequence $\{A_n\}$ and we write $\lim_{n\to\infty} A_n = A$.

Definition 2: A multivalued mapping (or set valued mapping) F on X into X is a point to set correspondence $x \to Fx$ such that Fx is a non-empty bounded subset of X for each $x \in X$. We denote such a mapping by $F: X \to B(X)$ (or CB(X)).

Definition 3: A multivalued map $F : X \to B(X)$ is said to be *continuous* at $x \in X$ if $x_n \to x$ in X implies $Fx_n \to Fx$ in B(X). F is continuous on X if F is continuous at every point of X.

An orbit of F at a point $x_0 \in X$ is a sequence $\{x_n\}$ in X given by

$$O(F, x_0) = \{x_n : x_n \in Fx_{n-1}, \quad n = 1, 2, 3, \ldots\}$$

Definition 4: A metric space X is said to be F-orbitally complete if every Cauchy sequence which is a subsequence of an orbit of F at each point $x \in X$ converges to a point of X.

Definition 5: A single valued mapping T of X into X is *orbitally continuous* on X if for each $x \in X$,

$$\lim_{n\to\infty} T^{n_i} x = u \quad \text{implies} \quad \lim_{n\to\infty} T(T^{n_i} x) = Tu.$$

Definition 6: A point $x \in X$ is said to be a fixed point of a multivalued map $F: X \to B(X)$ is $x \in F(X)$

The following fixed point theorem was proved by Achari[1] for Ciric type maps[3].

Theorem A (Achari[1]): Let X be an T-orbitally complete metric space and T be an orbitally continuous self-mapping of X satisfying

(A) min {d(Tx, Ty) d(x, y), d (x, Tx) d(y, Ty)} – min {d(x, Tx) d(x, Ty), d (y, Ty) d(y, Tx)} $\leq$ q d(x, y) min {d(x, Tx), d(y, Ty)}

for all x, y $\in$ X, $0 < q < 1$, d(x, Tx) $\neq 0$ and d(y, Ty) $\neq 0$.

Then for each x $\in$ X, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Using the technique of Taskovic [16], Jain and Bohre [5] generalized the above result as follows:

Theorem B (Jain and Bohre[5]): Let X be an F-orbitally complete metric space and T be an orbitally continuous self-mapping of X satisfying

(B) α_1 d(Tx, Ty) d(x, y) + α_2 d(x, Tx) d(y, Ty) – min {d(x, Tx) d(x, Ty), d(y, Ty) d(y, Tx)} $\leq$ β d(x, y) min {d(x, Tx), d(y, Ty)}

for all x, y $\in$ X, d(x, Tx) $\neq 0$ and d(y, Ty) $\neq 0$, where α_1, α_2 and β are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$. Then for each x $\in$ X, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

RESULTS: We prove the following:

Theorem: Let X be F-orbitally complete metric space and F: X $\to$ B(X) be continuous mapping satisfying

(1) $\alpha_1 \delta(\overline{F}x, \overline{F}y)^r d(x, y) + \alpha_2 \delta(x, \overline{F}x) \delta(y, \overline{F}y)^r$ – min{$d(x, \overline{F}x)$ $d(x, \overline{F}y)$, $d(y, \overline{F}y)^r d(y, \overline{F}x)$} $\leq \beta\, d(x, y)\, d(y, \overline{F}y)^{r-1}$ min {$d(x, \overline{F}x)$, $d(y, \overline{F}y)$}

for all x, y $\in$ X, where $r \geq 1$ is an integer, $d(x, \overline{F}x) \neq 0$ and $d(y, \overline{F}y) \neq 0$, α_1, α_2 and β are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$, then there exists x $\in$ X such that x $\in \overline{F}x$ where $\overline{F}$ denotes the closure of F. If F is a point closed mapping, then F has fixed point.

Proof: Let $x_o \in X$ be an arbitrary point is X. Define sequence $\{x_n\}$ in X by

$$x_1 \in \overline{F}x_0, \quad x_2 \in \overline{F}x_1, \ldots\ldots\ldots\ldots, \quad x_n \in \overline{F}x_{n-1}.$$

Let us suppose that $d(x_n, \overline{F}x_n) > 0$ for all n = 0, 1, 2, ….. (Otherwise for some positive integer n, $x_n \in \overline{F}x_n$).

Applying the condition(1) for $x = x_{n-1}$ and $y = x_n$, we have;

$\alpha_1 \delta(\overline{F}x_{n-1}, \overline{F}x_n)^r d(x_{n-1}, x_n) + \alpha_2 \delta(x_{n-1}, \overline{F}x_{n-1}) \delta(x_n, \overline{F}x_n)^r$ – min {$d(x_{n-1}, \overline{F}x_{n-1}) d(x_{n-1}, \overline{F}x_n)$, $d(x_n, \overline{F}x_n) d(x_n, \overline{F}x_{n-1})$}$\leq \beta\, d(x_{n-1}, x_n)\, d(x_n, \overline{F}x_n)^{r-1}$ min {$d(x_{n-1}, \overline{F}x_{n-1})$, $d(x_n, \overline{F}x_n)$}

or, $\alpha_1 d(x_n, x_{n+1})^r d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) d(x_n, x_{n+1})^r$ – min {$d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1})$, $d(x_n, x_{n+1})^r d(x_n, x_n)$} $\leq \beta\, d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}$ min{$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$}

or, $(\alpha_1 + \alpha_2) d(x_n, x_{n+1})^r d(x_{n-1}, x_n)$ - min {$d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1})$, 0} $\leq \beta\, d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}$ min{$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$}

or, $(\alpha_1 + \alpha_2) d(x_n, x_{n+1})^r d(x_{n-1}, x_n) \leq \beta\, d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}$ min {$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$}

or, $(\alpha_1 + \alpha_2) d(x_n, x_{n+1})^r \leq \beta\, d(x_n, x_{n+1})^{r-1}$ min {$d(x_{n-1}, x_n) d(x_n, x_{n+1})$}

or, $(\alpha_1 + \alpha_2) d(x_n, x_{n+1}) \leq \beta$ min {$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$}

or, $d(x_n, x_{n+1}) \leq \dfrac{\beta}{(\alpha_1 + \alpha_2)}$ min {$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$} = k min{$d(x_{n-1}, x_n)$, $d(x_n, x_{n+1})$}

where;

$$k = \frac{\beta}{(\alpha_1 + \alpha_2)} < 1.$$

Now, if $d(x_{n-1}, x_n)$ is minimum, then we get; $d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$ and if $d(x_n, x_{n+1})$ is minimum, then we have; $d(x_n, x_{n+1}) \leq k\, d(x_n, x_{n+1})$ which is contradiction, since $k < 1$

So we obtain; $d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$.

Proceeding in this manner we obtain;

$d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \ldots\ldots \leq k^n d(x_0, x_1)$.

Since $0 < k < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in X and since X is orbitally complete, there is a point x $\in$ X such that $x_n \to x$. Now the continuity of F implies that $Fx_n \to Fx$ in B(X).

It remains to show that d(x, Fx) = 0 that is x $\in \overline{F}x$. Suppose y $\in \overline{F}x$, then for any n,

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

and therefore,

$$d(x, Fx) \leq d(x, x_n) + d(x_n, Fx).$$

Since $x_n \to x$, for given $\varepsilon > 0$ we can choose an N_1 such that $d(x_n, x) < {}^{\varepsilon}/_3$ for all $n \geq N_1$. On the other hand, since $Fx_n \to Fx$, for the same ε we can choose an N_2 such that

$$Fx_{n-1} \subset A_{\varepsilon/3} = \bigcup_{x \in Fx} S\left(a, \frac{\varepsilon}{3}\right)$$

for all $n-1 \geq N_2$. Further, since $x_n \in \overline{F}x_{n-1}$, there exists a y $\in Fx_{n-1}$ such that

$$d(x_n, y) < \frac{\varepsilon}{3} \text{ and } y \in Fx_{n-1} \subset \bigcup_{a \in Fx} S\left(a, \frac{\varepsilon}{3}\right)$$

Implies that there exists an a $\in$ Fx such that $d(a, y) < \dfrac{\varepsilon}{3}$. Thus;

$$d(x_n, Fx) \leq d(x_n, a) \leq d(x_n, y) + d(y, a)$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \frac{2}{3}\varepsilon,$$

for all $n-1 \geq N_2$. Let $N = \max\{N_1, N_2\}$.
Then;

$$d(x, Fx) \leq d(x, x_n) + d(x_n, Fx)$$
$$< \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon$$
$$= \varepsilon,$$

for all $n \geq N$ and so; $x \in \overline{F}x$, since ε is arbitrary.
If F is a point closed mapping, i.e. Fx is closed for each $x \in X$, then $x \in Fx$ and therefore F has a fixed point. This completes the proof of Theorem 1.

Remark: If F is a single valued mapping T, r = 1 in Theorem 1 it reduces to Theorem B.

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